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LETTER TO THE EDITOR

Random walks and self-avoiding walks on self-affine fractals

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Abstract. We discuss random walks and self-avoiding walks on a self-affine fractal which is a Cartesian product of two fractals. Mean-square distances on the two sub-fractals have different scalings. Flory-type formulae are derived.

Countless work has been done on the scale invariant objects, fractals [1–3]. Fractals usually studied are self-similar, i.e. invariant with respect to isotropic contraction or dilution. In many physically relevant cases, however, the structure of the objects is such that it is invariant only under direction-dependent scaling. These anisotropically scale invariant objects are called self-affine fractals. Single-valued nowhere-differentiable functions [1–3], such as the record of the one-dimensional random walker (the displacement versus time function), are typical examples. Such a function has the scaling properties $F(t) \sim b^{-H}F(bt)$, and is characterized by a non-trivial local fractal dimension $D_l = 2-H$, but the global dimension is almost unity. On the other hand, the Cartesian product of two fractals with different fractal dimensions d_1 and d_2 embedded into two independent D_1 and D_2-dimensional Euclidean spaces, respectively, could lead to a self-affine object with non-trivial global fractal properties [3]. The Cartesian product, or the direct product of two graphs, is defined as the following [4]: let $V(A)$ and $V(B)$ represent the vector sets of graphs A and B, then its vertex set $V(C)$ consists of all pairs (i, j) where $i \in V(A)$ and $j \in V(B)$. Adjacency on C should also be defined. Let vertices (i, j) and (k, l) be adjacent on C if either i is adjacent to k on A and $j = l$, or $i = k$ and j is adjacent to l on B. Hence, for example, the square lattice is a Cartesian product of two lines. Schwalm and Schwalm [4] suggested that a Cartesian product of two two-dimensional fractals could be a model for some disordered material. Since the product of two two-dimensional fractals cannot be defined geometrically in three-dimensional physical space, this suggested structure can only be understood, at most, to be defined by the Hamiltonian matrix elements. A variety of growth models leads to self-affine aggregates, which could be treated as the products of self-similar fractals. A typical example is the directed percolation cluster at criticality [5], with different fractal dimensions along and perpendicular to the composing directions, respectively. Another example is the product of a two-dimensional fractal and a linear chain [6, 7]. If the two-dimensional fractal is a Koch curve, this could be viewed as a model of landscape [2]. One could also manufacture a periodic fractal superlattice, which is the product of a two-dimensional fractal and a periodic chain.

Such a structure has been termed a 'bifractal'. According to the addition rule [1–3], the global fractal dimension of the bifractal is $d = d_1 + d_2$. This was tested numerically on critical directed percolation clusters [5]. We have studied the dynamical properties of

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such an object [8], showing the additivity of spectral dimensions [9], i.e. $\tilde{d} = \tilde{d}_1 + \tilde{d}_2$ where $\tilde{d}$, $\tilde{d}_1$ and $\tilde{d}_2$ are the global spectral dimension and spectral dimensions of sub-fractals, respectively.

Consider a random walk of total N steps on a bifractal lattice, the sub-fractals of which are perpendicular to each other. Then on average there are, respectively, $N_1 = z_1 N/(z_1+z_2)$ and $N_2 = z_2 N/(z_1 + z_2)$ steps on the sub-fractals, where z_1 and z_2 are, respectively, coordinate numbers in the sub-fractals. The random walk can thus be decomposed to independent N_1 and N_2 steps on sub-fractals. Hence the mean-square distances in the two sub-fractals behave asymptotically as

$$\langle R_i^2(N)\rangle \sim N^{2/d_w^{(i)}} \qquad i = 1, 2 \tag{1}$$

where $d_w^{(i)} = 2d_i/\tilde{d}_i$ are fractal dimensionalities of random walks on the two sub-fractals. Generally there is no scaling for the global mean-square distances $\langle R^2(N)\rangle = \langle R_1^2(N)\rangle + \langle R_2^2(N)\rangle$ since $d_w^{(1)} \neq d_w^{(2)}$. For a Cartesian product of Euclidean lattices, $d_i = \tilde{d}_i$, hence $d_w^{(i)} = 2$ is universal and there is a scaling for $\langle R^2(N)\rangle$ with the global fractal dimensionality of random walk being also 2.

The volume that a random walker reaches on the bifractal after N steps is $R_1^{d_1} R_2^{d_2} \sim N^{d_1/d_w^{(1)}} N^{d_2/d_w^{(2)}} = N^{(\tilde{d}_1+\tilde{d}_2)/2}$. The condition that a random walk is transient is therefore $\tilde{d} = \tilde{d}_1 + \tilde{d}_2 > 2$. Thus by constructing a bifractal, one may easily obtain a structure with $\tilde{d} > 2$, hence a continuous symmetry can be spontaneously broken on it [10], while $\tilde{d} < 2$ for the fractals usually studied.

As is well known for Euclidean spaces as well as fractals, it is naturally expected for a self-avoiding walk (SAW) on a bifractal that

$$\langle R_i^2(N)\rangle \sim N^{2\nu_i} \qquad i = 1, 2. \tag{2}$$

Due to the particular restriction for a SAW, ν_is will not be independent and are simply those of 'free' sub-fractals, as indicated in the following Flory-type formulae.

To derive the Flory-type formula for a SAW, one should write the free energy F of an N-step SAW as the sum of energetic terms and entropic terms. Now the volume that a SAW in the bifractal reaches after N steps is $R_1^{d_1} R_2^{d_2}$. Therefore the free energy is

$$F = A\frac{N^2}{R_1^{d_1} R_2^{d_2}} + [-\ln P(R_1, R_2, N)], \tag{3}$$

where A is a coefficient, $P(R_1, R_2, N)$ is the probability of a random walker reaching distances R_1 on one sub-fractal and R_2 on the other after N steps. In general, the function $P(R, N)$ for SAWs on a fractal is [11]

$$P(R, N) \sim \exp[-B(R^{d_w}/N)^{\alpha}] \tag{4}$$

where B is a coefficient, α is an exponent, which has not been well established. $\alpha = 1$ for Euclidean spaces. Consequently, in a bifractal

$$P(R_1, R_2, N) \sim \exp[-B_1(R_1^{d_w^{(1)}}/N)_1^{\alpha} - B_2(R_2^{d_w^{(2)}}/N)_2^{\alpha}]. \tag{5}$$

Minimization of (3) with respect to R_1 and R_2, by setting $\partial F/\partial R_1 = \partial F/\partial R_2 = 0$, yields

$$N^{2+\alpha_1} \sim R_1^{\alpha_1 d_w^{(1)} + d_1} R_2^{d_2} \tag{6}$$

and

$$N^{2+\alpha_2} \sim R_1^{d_1} R_2^{\alpha d_w^{(2)}+d_2}. \tag{7}$$

Substituting $R_1 \sim N^{\nu_1}$ and $R_2 \sim N^{\nu_2}$, one obtains finally

$$\nu_1 = \frac{(2+\alpha_1)\alpha_2 d_w^{(2)} + (\alpha_1 - \alpha_2)d_2}{\alpha_1\alpha_2 d_w^{(1)} d_w^{(2)} + \alpha_1 d_w^{(1)} d_2 + \alpha_2 d_w^{(2)} d_1} \tag{8}$$

and

$$\nu_2 = \frac{(2+\alpha_2)\alpha_1 d_w^{(1)} + (\alpha_2 - \alpha_1)d_1}{\alpha_1\alpha_2 d_w^{(1)} d_w^{(2)} + \alpha_1 d_w^{(1)} d_2 + \alpha_2 d_w^{(2)} d_1} \tag{9}$$

from which we know that there exists global scaling for mean-square distances ($\nu_1 = \nu_2$) when $d_w^{(1)} = d_w^{(2)}$ and $\alpha_1 = \alpha_2$.

To be more precise, the various dimensionalities in the above formulas should be replaced by those of the backbone of the lattice [11, 12], since SAWs can only move on the backbone, otherwise they would be trapped at the dangling ends. The following examples we discuss are their own backbone, so we neglect this matter.

We may treat the square lattice as a Cartesian product of two linear lines. Given $d_w^{(1)} = d_w^{(2)} = 2$, $\alpha_1 = \alpha_2 = 1$ and $d_1 = d_2 = 1$, we obtain from (8) and (9) that $\nu_1 = \nu_2 = 3/4$. Similarly, regarding the cubic lattice as a Cartesian product of a linear line and a square lattice, we obtain $\nu_1 = \nu_2 = 3/5$. These are well-known results. In fact, for a Cartesian product of Euclidean lattices, equations (8) and (9) reduce to

$$\nu_1 = \nu_2 = \frac{3}{2=d} \tag{10}$$

where $d = d_1 + d_2$ is the global dimension. Hence the global scaling exponent of mean-square distances can be defined, and the original Flory formula is recovered.

The best formula for α is argued in [11] to be

$$\alpha = \frac{d_{\min}}{d_w - d_{\min}} \tag{11}$$

where $d_{\min}$ is the fractal dimension of the minimum path on a fractal. For exact fractals, we may use the approximation [11, 13]

$$\alpha = \frac{1}{d_w - 1}. \tag{12}$$

This is used in the following examples of bifractals.

One is a periodic branching Koch lattice, which is periodic in one direction, while on planes perpendicular to it, the lattice is branching Koch curves with fractal dimension $d_2 = \ln 5/\ln 3$ and fractal dimension of random walks $d_w^{(2)} = \ln 5/\ln 3 + \ln(8/3)/\ln 3$ [12]. One easily obtains $\nu_1 \approx 0.687$ and $\nu_2 \approx 0.640$. If the Koch curves are non-branching, with fractal and spectral dimension, respectively, $\ln 4/\ln 3$ and 1, then $\nu_1 = 0.721$, $\nu_2 = 0.663$. For a periodic Sierpinski lattice, which is the product of a periodic chain and a two-dimensional Sierpinski gasket, $d_2 = \ln 3/\ln 2$, $d_w^{(2)} = \ln 5/\ln 2$ [12]. Therefore $\nu_1 \approx 0.670$, $\nu_2 \approx 0.624$.

It is interesting to extend our study to random walks and SAWs on directed percolation clusters, and compare the mean field-type results presented here with Monte Carlo and renormalization group approaches. However, we feel that the latter is difficult because of the anisotropy of bifractals.

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